

Evaluation of the non-elementary integral $\int e^{\lambda x^\alpha} dx$, $\alpha \geq 2$, and related integrals

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Abstract A formula for the non-elementary integral $\int e^{\lambda x^\alpha} dx$, where λ is complex and α is real and greater or equal two, is obtained in terms of the confluent hypergeometric function ${}_1F_1$. This result is verified by directly evaluating the area under the Gaussian Bell curve, corresponding to $\alpha = 2$, using the asymptotic expression for the confluent hypergeometric function and the Fundamental Theorem of Calculus (FTC).

Two different but equivalent expressions, one in terms of the confluent hypergeometric function ${}_1F_1$ and another one in terms of the hypergeometric function ${}_1F_2$, are obtained for each of these integrals, $\int \cosh(\lambda x^\alpha) dx$, $\int \sinh(\lambda x^\alpha) dx$, $\int \cos(\lambda x^\alpha) dx$ and $\int \sin(\lambda x^\alpha) dx$, $\lambda \in \mathbb{C}$, $\alpha \geq 2$. And the hypergeometric function ${}_1F_2$ is expressed in terms of the confluent hypergeometric function ${}_1F_1$.

Keywords Non-elementary integral · Hypergeometric function · Confluent hypergeometric function · Asymptotic evaluation · Fundamental theorem of calculus

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1 Introduction

Definition 1 An elementary function is a function of one variable built up using that variable and constants, together with a finite number of repeated algebraic operations and the taking of exponentials and logarithms [6].

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Definition 2 An indefinite integral is non-elementary if it can not be expressed in terms of elementary functions [6].

In 1835, Joseph Liouville established conditions in his theorem, known as Liouville 1835's Theorem [4,6], which can be used to determine whether or not an indefinite integral is elementary or non-elementary. Using Liouville 1835's Theorem, one can show that the indefinite integral $\int e^{\lambda x^\alpha} dx$, $\alpha \geq 2$, is non-elementary [4], and to my knowledge, there does not exist a formula for this non-elementary integral.

For instance, if $\alpha = 2$, $\lambda = -\beta^2 < 0$, where β is a real constant, the area under the Gaussian Bell curve can be calculated using double integration and then polar coordinates to obtain

$$\int_{-\infty}^{+\infty} e^{-\beta^2 x^2} dx = \frac{\sqrt{\pi}}{\beta}. \quad (1)$$

Is that possible to evaluate (1) by directly using the Fundamental Theorem of Calculus (FTC) as in equation (2)?

$$\int_{-\infty}^{+\infty} e^{-\beta^2 x^2} dx = \lim_{t \rightarrow -\infty} \int_t^0 e^{-\beta^2 x^2} dx + \lim_{t \rightarrow +\infty} \int_0^t e^{-\beta^2 x^2} dx. \quad (2)$$

The Central limit Theorem (CLT) in Probability theory [2] states that the probability that a random variable X does not exceed some observed value z is

$$P(X < z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{x^2}{2}} dx. \quad (3)$$

If the antiderivative of the function $g(x) = e^{\lambda x^2}$ is known, then one may choose to use the FTC to compute the cumulative probability $P(X < z)$ in (3) if the value of z is given or is known, instead of using numerical integration.

There are many other examples where the antiderivative of $g(x) = e^{\lambda x^\alpha}$, $\alpha \geq 2$ can be useful. For instance, using the FTC, formulas for integrals such as

$$\int_{-\infty}^x e^{t^{2n+1}} dt, x < \infty; \int_x^{\infty} e^{-t^{2n+1}} dt, x > -\infty; \int_{-\infty}^x t^{2n} e^{-t^2} dt, x \leq \infty, \quad (4)$$

where n is a positive integer, can be obtained if the antiderivative of $g(x) = e^{\lambda x^\alpha}$, $\alpha \geq 2$ is known.

In this paper, the antiderivative of $g(x) = e^{\lambda x^\alpha}$, $\alpha \geq 2$, is expressed in terms of a special function, the confluent hypergeometric ${}_1F_1$ [1]. And the confluent hypergeometric ${}_1F_1$ is an entire function [3] and its properties are well known [1,5].

The main goal here is to consider the most general case with λ complex ($\lambda \in \mathbb{C}$), evaluate the non-elementary integral $\int e^{\lambda x^\alpha} dx$, $\alpha \geq 2$, and thus make possible the use of the FTC to compute the definite integral

$$\int_A^B e^{\lambda x^\alpha} dx, \quad \lambda \in \mathbb{C}, \quad \alpha \geq 2, \quad (5)$$

for any A and B . And once (5) is evaluated, then integrals such as (1), (2), (3), (4) can also be evaluated.

Using the hyperbolic and Euler identities, $\cosh(\lambda x^\alpha) = (e^{\lambda x^\alpha} + e^{-\lambda x^\alpha})/2$, $\sinh(\lambda x^\alpha) = (e^{\lambda x^\alpha} - e^{-\lambda x^\alpha})/2$, $\cos(\lambda x^\alpha) = (e^{i\lambda x^\alpha} + e^{-i\lambda x^\alpha})/2$ and $\sin(\lambda x^\alpha) = (e^{i\lambda x^\alpha} - e^{-i\lambda x^\alpha})/(2i)$, the integrals $\int \cosh(\lambda x^\alpha) dx$, $\int \sinh(\lambda x^\alpha) dx$, $\int \sin(\lambda x^\alpha) dx$ and $\int \cos(\lambda x^\alpha) dx$, where $\lambda \in \mathbb{C}$ and $\alpha \geq 2$, are evaluated in terms of ${}_1F_1$. They are also expressed in terms of the hypergeometric ${}_1F_2$. And some expressions of the hypergeometric function ${}_1F_2$ in terms of the confluent hypergeometric function ${}_1F_1$ are therefore obtained.

2 Evaluation of $\int_A^B e^{\lambda x^\alpha} dx$

Proposition 1 *The function $G(x) = x {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; \lambda x^\alpha\right)$, where ${}_1F_1$ is a confluent hypergeometric function [1], λ is an arbitrarily constant and $\alpha \geq 2$, is the antiderivative of the function $g(x) = e^{\lambda x^\alpha}$. Thus,*

$$\int e^{\lambda x^\alpha} dx = x {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; \lambda x^\alpha\right) + C. \quad (6)$$

Proof We expand $g(x) = e^{\lambda x^\alpha}$ as a Taylor series and integrate the series term by term. We also use the Pochhammer's notation [1] for the gamma function, $\Gamma(\alpha + n) = \Gamma(\alpha)(\alpha)_n$, where $(\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1)$, and the property of the gamma function $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ [1]. For example, $\Gamma(n + \alpha + 1) = (n + \alpha)\Gamma(n + \alpha)$ for any real n . We then obtain

$$\begin{aligned} \int g(x) dx &= \int e^{\lambda x^\alpha} dx = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int x^{\alpha n} dx \\ &= \frac{x}{\alpha} \sum_{n=0}^{\infty} \frac{\Gamma\left(n + \frac{1}{\alpha}\right)}{\Gamma\left(n + \frac{1}{\alpha} + 1\right)} \frac{(\lambda x^\alpha)^n}{n!} + C \\ &= x \sum_{n=0}^{\infty} \frac{\left(\frac{1}{\alpha}\right)_n}{\left(\frac{1}{\alpha} + 1\right)_n} \frac{(\lambda x^\alpha)^n}{n!} + C \\ &= x {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; \lambda x^\alpha\right) + C \\ &= G(x) + C. \end{aligned} \quad (7)$$

Example 1 One can now evaluate $\int x^{2n} e^{\lambda x^2} dx$ for any positive integer n . Using integration by parts,

$$\int x^{2n} e^{\lambda x^2} dx = \frac{x^{2n-1}}{2\lambda} e^{\lambda x^2} - \frac{2n-1}{2\lambda} \int x^{2n-2} e^{\lambda x^2} dx. \quad (8)$$

1. For $n = 1$,

$$\int x^2 e^{\lambda x^2} dx = \frac{x}{2\lambda} e^{\lambda x^2} - \frac{1}{2\lambda} \int e^{\lambda x^2} dx = \frac{x}{2\lambda} e^{\lambda x^2} - \frac{x}{2\lambda} {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; \lambda x^2\right) + C. \quad (9)$$

2. For $n = 2$,

$$\begin{aligned} \int x^4 e^{\lambda x^2} dx &= \frac{x^3}{2\lambda} e^{\lambda x^2} - \frac{3}{2\lambda} \int x^2 e^{\lambda x^2} dx \\ &= \frac{x^3}{2\lambda} e^{\lambda x^2} - \frac{3x}{4\lambda^2} e^{\lambda x^2} + \frac{3x}{4\lambda^2} {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; \lambda x^2\right) + C. \end{aligned} \quad (10)$$

Example 2 Using the method of integrating factors, the first-order ordinary differential equation,

$$y' + 2xy = 1, \quad (11)$$

has solution

$$y(x) = e^{-x^2} \left(\int e^{x^2} dx + C \right) = x e^{-x^2} {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; x^2\right) + C e^{-x^2}. \quad (12)$$

It is also important to note that the function $\operatorname{erf}(x)$,

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt = \frac{2x}{\sqrt{\pi}} {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -x^2\right), \quad (13)$$

is a particular solution of the first-order initial-value problem

$$\sqrt{\pi} y' = 2e^{-x^2}, x \geq 0, y(0) = 0. \quad (14)$$

It only takes positive values of x and corresponds to a particular case of Proposition 1 with $\lambda = -1$, $\alpha = 2$ and the constant $C = 0$.

Lemma 1 Assume that an antiderivative of $g(x) = e^{\lambda x^\alpha}$, $\lambda \in \mathbb{C}$ and $\alpha \geq 2$, exists and is $G(x)$.

1. If the real part of $\lambda < 0$ and α is even, then the limits $\lim_{x \rightarrow -\infty} G(x)$ and

$\lim_{x \rightarrow +\infty} G(x)$ are finite (constants). And thus the Lebesgue integral $\int_{-\infty}^{+\infty} |e^{\lambda x^\alpha}| dx < \infty$.

2. If λ is real, then the point $(0, G(0)) = (0, 0)$ is an inflection point of the curve $Y = G(x)$, $x \in \mathbb{R}$.

3. And if $\lambda < 0$ and real, and α is even, then the limits $\lim_{x \rightarrow -\infty} G(x)$ and $\lim_{x \rightarrow +\infty} G(x)$ are finite. And there exists a positive real constant $\theta > 0$ such that $\lim_{x \rightarrow -\infty} G(x) = -\theta$ and $\lim_{x \rightarrow +\infty} G(x) = \theta$.

Proof 1. If λ is complex, then the antiderivative $G(x)$ of the function $g(x)$ is an entire function on the whole complex plane \mathbb{C} . And if the real part of $\lambda < 0$ and α is even, then $\lim_{x \rightarrow \pm\infty} G'(x) = \lim_{x \rightarrow \pm\infty} g(x) = \lim_{x \rightarrow \pm\infty} e^{\lambda x^\alpha} = 0$. By Liouville theorem, $G(x)$ has to be constant as $x \rightarrow \pm\infty$ (section 3.1.3 in [3]). Let λ_r and λ_i be the real and imaginary parts of λ respectively. Then, the Lebesgue integral, $\int_{-\infty}^{+\infty} |e^{\lambda x^\alpha}| dx = \int_{-\infty}^{+\infty} |e^{(\lambda_r + i\lambda_i)x^\alpha}| dx = \int_{-\infty}^{+\infty} |e^{\lambda_r x^\alpha}| |e^{i\lambda_i x^\alpha}| dx = \int_{-\infty}^{+\infty} |e^{\lambda_r x^\alpha}| dx < \infty$ since $G(x)$ is constant as $x \rightarrow \pm\infty$.

Otherwise, if the real part of $\lambda < 0$ and α is odd, then $\lim_{x \rightarrow -\infty} g(x)$ diverges

and so does the integral $\int_{-\infty}^{+\infty} e^{\lambda x^\alpha} dx$. And if the real part of $\lambda > 0$, then

$\lim_{x \rightarrow -\infty} g(x)$ diverges and so does $\int_{-\infty}^{+\infty} e^{\lambda x^\alpha} dx$ regardless of the value of α .

2. At $x = 0$, $g(0) = 1$. And so, around $x = 0$, the antiderivative $G(x) \sim x$ because $G'(0) = g(0) = 1$. This gives $(0, G(0)) = (0, 0)$. Since, $G''(x) = g'(x) = \lambda \alpha x^{\alpha-1} e^{\lambda x^\alpha}$ for $\alpha \geq 2$, then $G''(0) = 0$. Hence, by the second derivative test, if λ is real, then the point $(0, G(0)) = (0, 0)$ is an inflection point of the curve $Y = G(x)$, $x \in \mathbb{R}$.
3. If λ is a real constant then $G(x)$ is analytic on \mathbb{R} . For $\lambda < 0$, $\lim_{x \rightarrow -\infty} g(x) = \lim_{x \rightarrow +\infty} g(x) = 0$. Therefore, by Liouville theorem, $G(x)$ has to be constant as

$x \rightarrow \pm\infty$ since if $\lambda \in \mathbb{C}$ and its real part is negative, then $\int_{-\infty}^{+\infty} |e^{\lambda x^\alpha}| dx < \infty$.

In addition, the fact that $G''(x) < 0$ if $x < 0$ and $G''(x) > 0$ if $x \geq 0$ implies that $G(x)$ is concave upward on the interval $(-\infty, 0)$ while is concave downward on the interval $(0, +\infty)$. Moreover, the fact that $g(x) = G'(x)$ is symmetric about the y -axis (even) implies that $G(x)$ has to be antisymmetric about the y -axis (odd). Hence there exists a positive constant $\theta > 0$ such that $\lim_{x \rightarrow -\infty} G(x) = -\theta$ while $\lim_{x \rightarrow +\infty} G(x) = \theta$. This completes the proof.

If $\lambda = -1$ and $\alpha = 2$, then

$$\int e^{-x^2} dx = x {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -x^2\right) + C. \quad (15)$$

According to (15), the antiderivative of $g(x) = e^{-x^2}$ is $G(x) = x {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -x^2\right)$. Its graph as a function of x , sketched using MATLAB, is shown in Figure 1,

it is in agreement with Lemma 1. It is actually seen in Figure 1 that $(0, 0)$ is an inflection point and that $G(x)$ reaches some constants as $x \rightarrow \pm\infty$ as predicted by Lemma 1.

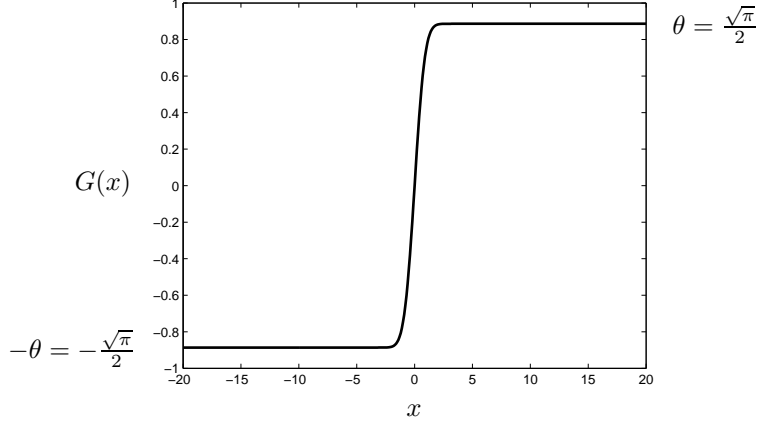


Fig. 1 Plot of the antiderivative of $g(x) = e^{-x^2}$ given by (15).

Lemma 2 Consider $G(x)$ in Proposition 1.

1. Then for $|x| \gg 1$,

$$G(x) = x {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; \lambda x^\alpha\right) \sim \begin{cases} \Gamma\left(\frac{1}{\alpha} + 1\right) \frac{e^{i\frac{\pi}{\alpha}}}{\lambda^{\frac{1}{\alpha}}} \frac{x}{|x|} + \frac{e^{\lambda x^\alpha}}{\alpha \lambda x^{\alpha-1}}, & \text{if } \alpha \text{ is even} \\ \Gamma\left(\frac{1}{\alpha} + 1\right) \frac{e^{i\frac{\pi}{\alpha}}}{\lambda^{\frac{1}{\alpha}}} + \frac{e^{\lambda x^\alpha}}{\alpha \lambda x^{\alpha-1}}, & \text{if } \alpha \text{ is odd} \end{cases} \quad (16)$$

2. Assume that $\alpha \geq 2$ is even and $\lambda < 0$, and set $\lambda = -\beta^2$, where β is a real number, preferably positive. Then,

$$\begin{aligned} G(-\infty) &= \lim_{x \rightarrow -\infty} G(x) \\ &= \lim_{x \rightarrow -\infty} x {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; -\beta^2 x^\alpha\right) = -\frac{1}{\beta^{\frac{2}{\alpha}}} \Gamma\left(\frac{1}{\alpha} + 1\right) \end{aligned} \quad (17)$$

and

$$\begin{aligned} G(+\infty) &= \lim_{x \rightarrow +\infty} G(x) \\ &= \lim_{x \rightarrow +\infty} x {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; -\beta^2 x^\alpha\right) = \frac{1}{\beta^{\frac{2}{\alpha}}} \Gamma\left(\frac{1}{\alpha} + 1\right). \end{aligned} \quad (18)$$

3. And by the FTC,

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-\beta^2 x^\alpha} dx &= G(+\infty) - G(-\infty) \\ &= \frac{1}{\beta^{\frac{2}{\alpha}}} \Gamma\left(\frac{1}{\alpha} + 1\right) - \left(-\frac{1}{\beta} \Gamma\left(\frac{1}{\alpha} + 1\right)\right) \\ &= \frac{2}{\beta^{\frac{2}{\alpha}}} \Gamma\left(\frac{1}{\alpha} + 1\right). \end{aligned} \quad (19)$$

Proof 1. To prove (16), we use the asymptotic series for the confluent hypergeometric function that is valid for $|z| \gg 1$ ([1], formula 13.5.1),

$$\begin{aligned} \frac{{}_1F_1(a; b; z)}{\Gamma(b)} &= \frac{e^{\pm i\pi a} z^{-a}}{\Gamma(b-a)} \left\{ \sum_{n=0}^{R-1} \frac{(a)_n (1+a-b)_n}{n!} (-z)^{-n} + O(|z|^{-R}) \right\} \\ &\quad + \frac{e^z z^{a-b}}{\Gamma(a)} \left\{ \sum_{n=0}^{S-1} \frac{(b-a)_n (1-a)_n}{n!} (z)^{-n} + O(|z|^{-S}) \right\}, \end{aligned} \quad (20)$$

where a and b are constants, and the upper sign being taken if $-\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}$ and the lower sign if $-\frac{3\pi}{2} < \arg(z) \leq -\frac{\pi}{2}$.

We set $z = \lambda x^\alpha$, $a = \frac{1}{\alpha}$ and $b = \frac{1}{\alpha} + 1$, and obtain

$$\begin{aligned} \frac{{}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; \lambda x^\alpha\right)}{\Gamma\left(\frac{1}{\alpha} + 1\right)} &= \frac{e^{i\frac{\pi}{\alpha}}}{(\lambda x^\alpha)^{\frac{1}{\alpha}}} \left\{ \sum_{n=0}^{R-1} \frac{\left(\frac{1}{\alpha}\right)_n}{n!} (\lambda x^\alpha)^{-n} + O(\lambda x^\alpha)^{-R} \right\} \\ &\quad + \frac{1}{\Gamma\left(\frac{1}{\alpha}\right)} \frac{e^{\lambda x^\alpha}}{\lambda x^\alpha} \left\{ \sum_{n=0}^{S-1} \frac{(1)_n \left(\frac{1}{\alpha}\right)_n}{n!} (\lambda x^\alpha)^{-n} + O(\lambda x^\alpha)^{-S} \right\}. \end{aligned} \quad (21)$$

Then, for $|x| \gg 1$,

$$\begin{aligned} \frac{e^{i\frac{\pi}{\alpha}}}{(\lambda x^\alpha)^{\frac{1}{\alpha}}} \left\{ \sum_{n=0}^{R-1} \frac{\left(\frac{1}{\alpha}\right)_n}{n!} (\lambda x^\alpha)^{-n} + O(\lambda x^\alpha)^{-R} \right\} \\ \sim \begin{cases} \frac{e^{i\frac{\pi}{\alpha}}}{\lambda^{\frac{1}{\alpha}}} \frac{1}{|x|}, & \text{if } \alpha \text{ is even} \\ \frac{e^{i\frac{\pi}{\alpha}}}{\lambda^{\frac{1}{\alpha}}} \frac{1}{x}, & \text{if } \alpha \text{ is odd} \end{cases} \end{aligned} \quad (22)$$

while

$$\frac{1}{\Gamma\left(\frac{1}{\alpha}\right)} \frac{e^{\lambda x^\alpha}}{\lambda x^\alpha} \left\{ \sum_{n=0}^{S-1} \frac{(1)_n \left(\frac{1}{\alpha}\right)_n}{n!} (\lambda x^\alpha)^{-n} + O(\lambda x^\alpha)^{-S} \right\} \sim \frac{1}{\Gamma\left(\frac{1}{\alpha}\right)} \frac{e^{\lambda x^\alpha}}{\lambda x^\alpha}. \quad (23)$$

Therefore for $|x| \gg 1$,

$${}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; \lambda x^\alpha\right) \sim \begin{cases} \Gamma\left(\frac{1}{\alpha} + 1\right) \frac{e^{i\frac{\pi}{\alpha}}}{\lambda^{\frac{1}{\alpha}}} \frac{1}{|x|} + \frac{e^{\lambda x^\alpha}}{\alpha \lambda x^\alpha}, & \text{if } \alpha \text{ is even} \\ \Gamma\left(\frac{1}{\alpha} + 1\right) \frac{e^{i\frac{\pi}{\alpha}}}{\lambda^{\frac{1}{\alpha}}} \frac{1}{x} + \frac{e^{\lambda x^\alpha}}{\alpha \lambda x^\alpha}, & \text{if } \alpha \text{ is odd} \end{cases} \quad (24)$$

Hence for $|x| \gg 1$,

$$G(x) = x {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; \lambda x^\alpha\right) \sim \begin{cases} \Gamma\left(\frac{1}{\alpha} + 1\right) \frac{e^{i\frac{\pi}{\alpha}}}{\lambda^{\frac{1}{\alpha}}} \frac{x}{|x|} + \frac{e^{\lambda x^\alpha}}{\alpha \lambda x^{\alpha-1}}, & \text{if } \alpha \text{ is even} \\ \Gamma\left(\frac{1}{\alpha} + 1\right) \frac{e^{i\frac{\pi}{\alpha}}}{\lambda^{\frac{1}{\alpha}}} + \frac{e^{\lambda x^\alpha}}{\alpha \lambda x^{\alpha-1}}, & \text{if } \alpha \text{ is odd} \end{cases} \quad (25)$$

2. Setting $\lambda = -\beta^2$, where β is real and positive and using (16), then for α even,

$$G(x) = x {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; -\beta^2 x^\alpha\right) \sim \frac{1}{\beta^{\frac{2}{\alpha}}} \Gamma\left(\frac{1}{\alpha} + 1\right) \frac{x}{|x|} - \frac{1}{\beta^2 \Gamma\left(\frac{1}{\alpha}\right)} \frac{e^{-\beta^2 x^\alpha}}{x^{\alpha-1}}, \quad (26)$$

Hence,

$$G(-\infty) = \lim_{x \rightarrow -\infty} x {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; -\beta^2 x^\alpha\right) = -\frac{1}{\beta^{\frac{2}{\alpha}}} \Gamma\left(\frac{1}{\alpha} + 1\right), \quad (27)$$

while

$$G(+\infty) = \lim_{x \rightarrow +\infty} x {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; -\beta^2 x^\alpha\right) = \frac{1}{\beta^{\frac{2}{\alpha}}} \Gamma\left(\frac{1}{\alpha} + 1\right). \quad (28)$$

3. By the FTC,

$$\begin{aligned} \int_{-\infty}^{+\infty} e^{-\beta^2 x^\alpha} dx &= \lim_{y \rightarrow -\infty} \int_y^0 e^{-\beta^2 x^\alpha} dx + \lim_{y \rightarrow +\infty} \int_0^y e^{-\beta^2 x^\alpha} dx \\ &= \lim_{y \rightarrow +\infty} y {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; -\beta^2 y^\alpha\right) - \lim_{y \rightarrow -\infty} y {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; -\beta^2 y^\alpha\right) \\ &= G(+\infty) - G(-\infty) \\ &= \frac{1}{\beta^{\frac{2}{\alpha}}} \Gamma\left(\frac{1}{\alpha} + 1\right) - \left(-\frac{1}{\beta^{\frac{2}{\alpha}}} \Gamma\left(\frac{1}{\alpha} + 1\right)\right) \\ &= \frac{2}{\beta^{\frac{2}{\alpha}}} \Gamma\left(\frac{1}{\alpha} + 1\right). \end{aligned} \quad (29)$$

We now verify whether (29) is correct or not by double integration. We first observe that (29) is valid for all even $\alpha \geq 2$. And so, if (29) is verified for $\alpha = 2$, we are done since (29) is valid for all even $\alpha \geq 2$. Considering $\alpha = 2$,

and setting $\alpha = 2$ in (29) gives

$$\begin{aligned}
 \int_{-\infty}^{+\infty} e^{-\beta^2 x^2} dx &= \lim_{y \rightarrow -\infty} \int_y^0 e^{-\beta^2 x^2} dx + \lim_{y \rightarrow +\infty} \int_0^y e^{-\beta^2 x^2} dx \\
 &= \lim_{y \rightarrow +\infty} y {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -\beta^2 y^2\right) - \lim_{y \rightarrow -\infty} y {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -\beta^2 y^2\right) \\
 &= G(+\infty) - G(-\infty) = \frac{2}{\beta} \Gamma\left(\frac{3}{2}\right) = \frac{2}{\beta} \left(\frac{\pi}{2}\right) = \frac{\sqrt{\pi}}{\beta}. \quad (30)
 \end{aligned}$$

On the other hand,

$$\left(\int_{-\infty}^{\infty} e^{-\beta^2 x^2} dx \right)^2 = \left(\int_{-\infty}^{\infty} e^{-\beta^2 x^2} dx \right) \left(\int_{-\infty}^{\infty} e^{-\beta^2 y^2} dy \right) \quad (31)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\beta^2 (x^2 + y^2)} dy dx. \quad (32)$$

In polar coordinate,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\beta^2 (x^2 + y^2)} dy dx = \int_0^{2\pi} \int_0^{\infty} e^{-\beta^2 r^2} r dr d\theta = \frac{1}{2\beta^2} \int_0^{2\pi} d\theta = \frac{\pi}{\beta^2}. \quad (33)$$

Hence,

$$\int_{-\infty}^{\infty} e^{-\beta^2 x^2} dx = \sqrt{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2 + y^2)} dy dx} = \frac{\sqrt{\pi}}{\beta} \quad (34)$$

as before. This completes the proof.

Example 3 Setting $\lambda = -1$ and $\alpha = 2$ in Lemma 2 gives

$$\lim_{x \rightarrow -\infty} G(x) = \lim_{x \rightarrow -\infty} x {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -x^2\right) = -\frac{\sqrt{\pi}}{2} \quad (35)$$

and

$$\lim_{x \rightarrow +\infty} G(x) = \lim_{x \rightarrow -\infty} x {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -x^2\right) = \frac{\sqrt{\pi}}{2}. \quad (36)$$

This implies $\theta = \frac{\sqrt{\pi}}{2}$ in Lemma 1. And this is exactly the value of $G(x)$ as $x \rightarrow \infty$ in Figure 1. And also $\lim_{x \rightarrow -\infty} G(x) = -\theta = -\frac{\sqrt{\pi}}{2}$ as in Figure 1.

The FTC gives

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = G(+\infty) - G(-\infty) = \frac{\sqrt{\pi}}{2} - \left(-\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi}, \quad (37)$$

$$\int_0^{+\infty} e^{-x^2} dx = G(+\infty) - G(0) = \frac{\sqrt{\pi}}{2} - 0 = \frac{\sqrt{\pi}}{2} = \frac{1}{2} \int_{-\infty}^{+\infty} e^{-x^2} dx \quad (38)$$

and

$$\int_{-\infty}^0 e^{-x^2} dx = G(0) - G(-\infty) = 0 - \left(-\frac{\sqrt{\pi}}{2}\right) = \frac{\sqrt{\pi}}{2}. \quad (39)$$

Example 4 In this example, the integral

$$\int_{-\infty}^x e^{t^{2n+1}} dt, x < \infty, \quad (40)$$

where n is a positive integer, is evaluated using Proposition 1 and the asymptotic expression (16). Setting $\lambda = 1$ and $\alpha = 2n+1$ in Proposition 1, and using (16) gives

$$\begin{aligned} \int_{-\infty}^x e^{t^{2n+1}} dt &= \lim_{y \rightarrow -\infty} \int_y^x e^{t^{2n+1}} dt \\ &= x {}_1F_1\left(\frac{1}{2n+1}; \frac{2n+2}{2n+1}; x^{2n+1}\right) - \lim_{y \rightarrow -\infty} y {}_1F_1\left(\frac{1}{2n+1}; \frac{2n+2}{2n+1}; y^{2n+1}\right) \\ &= x {}_1F_1\left(\frac{1}{2n+1}; \frac{2n+2}{2n+1}; x^{2n+1}\right) - \Gamma\left(\frac{2n+2}{2n+1}\right), \quad x < \infty. \end{aligned} \quad (41)$$

One can also obtain,

$$\begin{aligned} \int_x^{\infty} e^{-t^{2n+1}} dt &= \lim_{y \rightarrow \infty} \int_x^y e^{-t^{2n+1}} dt \\ &= \lim_{y \rightarrow \infty} y {}_1F_1\left(\frac{1}{2n+1}; \frac{2n+2}{2n+1}; -y^{2n+1}\right) - x {}_1F_1\left(\frac{1}{2n+1}; \frac{2n+2}{2n+1}; -x^{2n+1}\right) \\ &= \Gamma\left(\frac{2n+2}{2n+1}\right) - x {}_1F_1\left(\frac{1}{2n+1}; \frac{2n+2}{2n+1}; -x^{2n+1}\right), \quad x > -\infty. \end{aligned} \quad (42)$$

Theorem 1 For any A and B , the FTC gives

$$\int_A^B e^{\lambda x^\alpha} dx = G(B) - G(A), \quad (43)$$

where G is the antiderivative of the function $g(x) = e^{\lambda x^\alpha}$ and is given in Proposition 1. And λ is any complex or real constant, and $\alpha \geq 2$.

Proof $G(x) = x {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; \lambda x^\alpha\right)$, where λ is any constant, is the antiderivative of $g(x) = e^{\lambda x^\alpha}$, $\alpha \geq 2$ by Proposition 1, Lemma 1 and Lemma 2.

And since the FTC works for $A = -\infty$ and $B = +\infty$ in (37), $A = 0$ and $B = +\infty$ in (38) and $A = -\infty$ and $B = 0$ in (39) by Lemma 2 if $\lambda = -1$ and $\alpha = 2$, and for all $\lambda < 0$ and all even $\alpha \geq 2$, then it has to work for other values of $A, B \in \mathbb{R}$ and for any $\lambda \in \mathbb{C}$ and $\alpha \geq 2$. This completes the proof.

Example 5 In this example, we apply Theorem 1 to the Central Limit Theorem in Probability theory [2]. The normal zero-one distribution of a random variable X is the measure $\mu(dx) = g_X(x)dx$, where dx is the Lebesgue measure and the function $g_X(x)$ is the probability density function (p.d.f) of the normal zero-one distribution [2], and is

$$g_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < +\infty. \quad (44)$$

A comparison with the function $g(x)$ in Proposition 1 and Lemma 2 gives $\lambda = -\beta^2 = -1/2$ and $\alpha = 2$. By Theorem 1, the cumulative probability, $P(X < z)$, is then given by

$$\begin{aligned} P(X < z) &= \mu\{(-\infty, z)\} = \int_{-\infty}^z g_X(x) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{x^2}{2}} dx \\ &= \frac{1}{2} + \frac{z}{\sqrt{2\pi}} {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -\frac{z^2}{2}\right). \end{aligned} \quad (45)$$

One may also use Theorem 1 to compute the probabilities, $P(-2 < X < 2) = \mu\{(-2, 2)\} = 0.4772 - (-0.4772) = 0.9544$, $P(-1 < X < 2) = \mu\{(-1, 2)\} = 0.4772 - (-0.3413) = 0.8185$ and so on.

3 Other related non-elementary integrals

Proposition 2 For $\alpha \geq 2$, the function $G(x) = x {}_1F_2\left(\frac{1}{2\alpha}; \frac{1}{2}, \frac{1}{2\alpha} + 1; \frac{\lambda^2 x^{2\alpha}}{4}\right)$, where ${}_1F_2$ is a hypergeometric function [1] and λ is an arbitrarily constant, is the antiderivative of the hyperbolic function $g(x) = \cosh(\lambda x^\alpha)$. Thus,

$$\int \cosh(\lambda x^\alpha) dx = x {}_1F_2\left(\frac{1}{2\alpha}; \frac{1}{2}, \frac{1}{2\alpha} + 1; \frac{\lambda^2 x^{2\alpha}}{4}\right) + C. \quad (46)$$

Proof

$$\begin{aligned}
\int g(x)dx &= \int \cosh(\lambda x^\alpha)dx \\
&= \int \sum_{n=0}^{\infty} \frac{(\lambda x^\alpha)^{2n}}{(2n)!} dx \\
&= \frac{x}{2\alpha} \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} \frac{x^{2\alpha n}}{n + \frac{1}{2\alpha}} + C \\
&= \frac{x}{2\alpha} \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2\alpha})}{\Gamma(2n+1)\Gamma(n + \frac{1}{2\alpha} + 1)} (\lambda^2 x^{2\alpha})^n + C \\
&= x \sum_{n=0}^{\infty} \frac{(\frac{1}{2\alpha})_n}{(\frac{1}{2})_n (\frac{1}{2\alpha} + 1)_n} \frac{\left(\frac{\lambda^2 x^{2\alpha}}{4}\right)^n}{n!} + C \\
&= x {}_1F_2\left(\frac{1}{2\alpha}; \frac{1}{2}, \frac{1}{2\alpha} + 1; \frac{\lambda^2 x^{2\alpha}}{4}\right) + C = G(x) + C. \quad (47)
\end{aligned}$$

Proposition 3 For $\alpha \geq 2$, the function $G(x) = \frac{\lambda x^{\alpha+1}}{\alpha+1} {}_1F_2\left(\frac{1}{2\alpha} + \frac{1}{2}; \frac{3}{2}, \frac{1}{2\alpha} + \frac{3}{2}; \frac{\lambda^2 x^{2\alpha}}{4}\right)$, where ${}_1F_2$ is a hypergeometric function [1] and λ is an arbitrarily constant, is the antiderivative of the hyperbolic function $g(x) = \sinh(\lambda x^\alpha)$. Thus,

$$\int \sinh(\lambda x^\alpha)dx = \frac{\lambda x^{\alpha+1}}{\alpha+1} {}_1F_2\left(\frac{1}{2\alpha} + \frac{1}{2}; \frac{3}{2}, \frac{1}{2\alpha} + \frac{3}{2}; \frac{\lambda^2 x^{2\alpha}}{4}\right) + C. \quad (48)$$

Proof

$$\begin{aligned}
\int g(x)dx &= \int \sinh(\lambda x^\alpha)dx \\
&= \int \sum_{n=0}^{\infty} \frac{(\lambda x^\alpha)^{2n+1}}{(2n+1)!} dx \\
&= \frac{\lambda x^{\alpha+1}}{2\alpha} \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n+1)!} \frac{x^{2\alpha n}}{n + \frac{1}{2\alpha} + \frac{1}{2}} + C \\
&= \frac{\lambda x^{\alpha+1}}{2\alpha} \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2\alpha} + \frac{1}{2})}{\Gamma(2n+2)\Gamma(n + \frac{1}{2\alpha} + \frac{3}{2})} (\lambda^2 x^{2\alpha})^n + C \\
&= \frac{\lambda x^{\alpha+1}}{\alpha+1} \sum_{n=0}^{\infty} \frac{(\frac{1}{2\alpha} + \frac{1}{2})_n}{(\frac{3}{2})_n (\frac{1}{2\alpha} + \frac{3}{2})_n} \frac{\left(\frac{\lambda^2 x^{2\alpha}}{4}\right)^n}{n!} + C \\
&= \frac{\lambda x^{\alpha+1}}{\alpha+1} {}_1F_2\left(\frac{1}{2\alpha} + \frac{1}{2}; \frac{3}{2}, \frac{1}{2\alpha} + \frac{3}{2}; \frac{\lambda^2 x^{2\alpha}}{4}\right) + C = G(x) + C. \quad (49)
\end{aligned}$$

We also can show as above that

$$\int \cos(\lambda x^\alpha) dx = x {}_1F_2\left(\frac{1}{2\alpha}; \frac{1}{2}, \frac{1}{2\alpha} + 1; -\frac{\lambda^2 x^{2\alpha}}{4}\right) + C \quad (50)$$

and

$$\int \sin(\lambda x^\alpha) dx = \frac{\lambda x^{\alpha+1}}{\alpha+1} {}_1F_2\left(\frac{1}{2\alpha} + \frac{1}{2}; \frac{3}{2}, \frac{1}{2\alpha} + \frac{3}{2}; -\frac{\lambda^2 x^{2\alpha}}{4}\right) + C, \quad (51)$$

Theorem 2 For any constants α and λ ,

1.

$$\begin{aligned} & {}_1F_2\left(\frac{1}{2\alpha}; \frac{1}{2}, \frac{1}{2\alpha} + 1; \frac{\lambda^2 x^{2\alpha}}{4}\right) \\ &= \frac{1}{2} \left[{}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; \lambda x^\alpha\right) + {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; -\lambda x^\alpha\right) \right]. \end{aligned} \quad (52)$$

2. And

$$\begin{aligned} & {}_1F_2\left(\frac{1}{2\alpha}; \frac{1}{2}, \frac{1}{2\alpha} + 1; -\frac{\lambda^2 x^{2\alpha}}{4}\right) \\ &= \frac{1}{2} \left[{}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; i\lambda x^\alpha\right) + {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; -i\lambda x^\alpha\right) \right]. \end{aligned} \quad (53)$$

Proof 1. Using Proposition 1, we obtain

$$\begin{aligned} \int \cosh(\lambda x^\alpha) dx &= \int \frac{e^{\lambda x^\alpha} + e^{-\lambda x^\alpha}}{2} dx \\ &= \frac{x}{2} \left[{}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; \lambda x^\alpha\right) + {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; -\lambda x^\alpha\right) \right] + C, \end{aligned} \quad (54)$$

Hence, comparing (47) with (54) gives (52).

2. On the other hand,

$$\begin{aligned} \int \cos(\lambda x^\alpha) dx &= \int \frac{e^{i\lambda x^\alpha} + e^{-i\lambda x^\alpha}}{2} dx \\ &= \frac{x}{2} \left[{}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; i\lambda x^\alpha\right) + {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; -i\lambda x^\alpha\right) \right] + C, \end{aligned} \quad (55)$$

Hence, comparing (50) with (55) gives (53).

Theorem 3 For any constants α and λ ,

1.

$$\begin{aligned} & \frac{\lambda x^\alpha}{\alpha+1} {}_1F_2\left(\frac{1}{2\alpha} + \frac{1}{2}; \frac{3}{2}, \frac{1}{2\alpha} + \frac{3}{2}; -\frac{\lambda^2 x^{2\alpha}}{4}\right) \\ &= \frac{1}{2} \left[{}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; \lambda x^\alpha\right) - {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; \lambda x^\alpha\right) \right]. \end{aligned} \quad (56)$$

2. And

$$\begin{aligned} & \frac{\lambda x^\alpha}{\alpha+1} {}_1F_2\left(\frac{1}{2\alpha} + \frac{1}{2}; \frac{3}{2}, \frac{1}{2\alpha} + \frac{3}{2}; -\frac{\lambda^2 x^{2\alpha}}{4}\right) \\ &= \frac{1}{2i} \left[{}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; i\lambda x^\alpha\right) - {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; -i\lambda x^\alpha\right) \right]. \end{aligned} \quad (57)$$

Proof 1. Using Proposition 1, we obtain

$$\begin{aligned} \int \sinh(\lambda x^\alpha) dx &= \int \frac{e^{\lambda x^\alpha} - e^{-\lambda x^\alpha}}{2} dx \\ &= \frac{x}{2} \left[{}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; \lambda x^\alpha\right) - {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; -\lambda x^\alpha\right) \right]. \end{aligned} \quad (58)$$

Hence, comparing (49) with (58) gives (56).

2. On the other hand,

$$\begin{aligned} \int \sin(\lambda x^\alpha) dx &= \int \frac{e^{i\lambda x^\alpha} - e^{-i\lambda x^\alpha}}{2i} dx \\ &= \frac{x}{2i} \left[{}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; i\lambda x^\alpha\right) - {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; -i\lambda x^\alpha\right) \right]. \end{aligned} \quad (59)$$

Hence, comparing (51) with (59) gives (57).

4 Conclusion

The non-elementary $\int e^{\lambda x^\alpha} dx$, where λ is an arbitrary constant and $\alpha \geq 2$, was expressed in term of the confluent hypergeometric function ${}_1F_1$. And using the properties of the confluent hypergeometric function ${}_1F_1$, the asymptotic expression for $|x| \gg 1$ of this integral was derived too. As established in Theorem 1, the definite integral (5) can now be computed using the FTC. For example, one can evaluate the area under the Gaussian Bell curve using the FTC rather than using double integration and then polar coordinates. One can also choose to use Theorem 1 to compute the cumulative probability for the normal distribution.

On one hand, the integrals $\int \cosh(\lambda x^\alpha) dx$, $\int \sinh(\lambda x^\alpha) dx$ and $\int \sin(\lambda x^\alpha) dx$ and $\int \cos(\lambda x^\alpha) dx$, $\lambda \in \mathbb{C}$, $\alpha \geq 2$ were evaluated in terms of ${}_1F_1$, while on another hand, they were expressed in terms of the hypergeometric ${}_1F_2$. This allowed to express the hypergeometric function ${}_1F_2$ in terms of the confluent hypergeometric function ${}_1F_1$.

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